Unique equilibrium states for geodesic flows in nonpositive curvature

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Fractal Geometry, Hyperbolic Dynamics and Thermodynamical Formalism

Joint work with K. Burns, V. Climenhaga, and D. Thompson

Outline

Topological entropy

- $\mathcal{F} = f_t : X \to X$ a smooth flow on a compact manifold
- Bowen ball : $B_T(x;\epsilon) = \{y : d(f_t y, f_t x) < \epsilon \text{ for } 0 \leq t \leq T\}$
- $x_1, \ldots, x_n$ are (T, ϵ)-spanning if $\bigcup_{i=1}^{n} B_T(x_i;\epsilon) = X$

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$\Lambda^{\text{span}}(T,\epsilon) = \inf\{\#(E) \mid E \subset X \text{ is } (T,\epsilon)\text{-spanning}\}$

$$h_{top}(\mathcal{F}) = \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log \Lambda^{\text{span}}(T,\epsilon)$$

Remark: An equivalent definition is $h_{top}(\mathcal{F}) = h_{top}(f_1)$ where the second term is the entropy of the time-1 map.

Measure entropy and the variational principle

For a flow $\mathcal{F} = (f_t)_{t\in\mathbb{R}}$ let $\mathcal{M}(f_t)$ be the set of f_t-invariant Borel probability measures and $\mathcal{M}(\mathcal{F}) = \bigcap_{t\in\mathbb{R}} \mathcal{M}(f_t)$ be the set of flow invariant Borel probability measures.

For $\mu \in \mathcal{M}(\mathcal{F})$ the measure theoretic entropy of $\mathcal{F}$ for μ is $h_\mu(\mathcal{F}) = h_\mu(f_1)$.

(**Variational Principle**) $h_{top}(\mathcal{F}) = \sup_{\mu\in\mathcal{M}(\mathcal{F})} h_\mu(\mathcal{F})$

Topological pressure

- $\varphi : X \to \mathbb{R}$ a continuous function. We will refer to this as a potential function or observable.

$$\Lambda^{\mathrm{span}}(\varphi; T, \epsilon) = \inf \left\{ \sum_{x \in E} e^{\int_0^T \varphi(f_t x)\, dt} \mid E \subset X \text{ is } (T, \epsilon)\text{-spanning} \right\}$$

$$P(\varphi, \mathcal{F}) = \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log \Lambda^{\mathrm{span}}(\varphi; T, \epsilon)$$

Remark: When $\varphi \equiv 0$, $P(\varphi, \mathcal{F}) = h_{top}(\mathcal{F})$

The variational principle for pressure

Let $\mu \in \mathcal{M}(\mathcal{F})$. Then

$$P_\mu(\varphi, \mathcal{F}) = h_\mu(\mathcal{F}) + \int \varphi \, d\mu$$

(**Variational Principle for Pressure**)

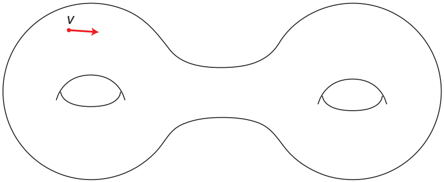
$$P(\varphi, \mathcal{F}) = \sup_{\mu \in \mathcal{M}(\mathcal{F})} P_\mu(\varphi, \mathcal{F})$$

$\mu \in \mathcal{M}(\mathcal{F})$ is an equilibrium state for φ if $P_\mu(\varphi, \mathcal{F}) = P(\varphi, \mathcal{F})$.

If the flow is C^∞ there is an equilibrium state for any continuous potential (Newhouse: upper semi continuity of $\mu \mapsto h_\mu$, and the set of measures is compact)

Geodesic flow

M compact Riemannian manifold with negative sectional curvatures.



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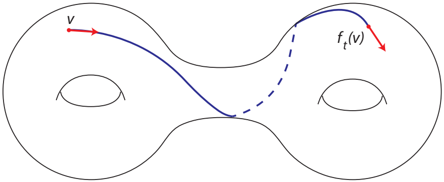
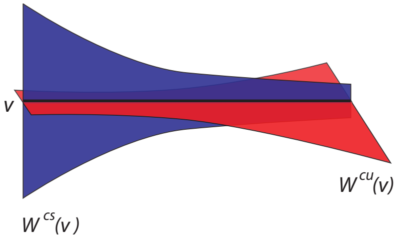


Let T^1M be the unit tangent bundle.
$f_t : T^1M \to T^1M$ geodesic flow

Properties of geodesic flows for negative curvature

- The flow is Anosov ($TT^1M = \mathbb{E}^s \oplus \mathbb{E}^c \oplus \mathbb{E}^u$) where the splitting is given by the flow direction ($\mathbb{E}^c$) and the tangent spaces to the stable and unstable horospheres ($\mathbb{E}^s$ and $\mathbb{E}^u$), and the flow is volume preserving (Liouville measure)



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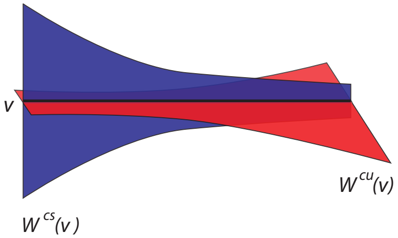

- Bowen: any Hölder continuous potential φ has a unique equilibrium state, and the geometric potential $\varphi^u = -\lim_{t\to 0} \frac{1}{t} \log \mathrm{Jac}(Df_t|E^u)$ has a unique equilibrium state

Outline

Nonpositive curvature

M compact manifold with nonpositive curvature

$f_t : T^1M \to T^1M$ geodesic flow. There are still foliations globally defined that are similar to stable and unstable foliations.

Two types of geodesic:

regular: horospheres in M do not make second order contact

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For a surface:
regular = geodesic passes through negative curvature
singular = curvature is zero everywhere along the geodesic

Rank of the manifold

A Jacobi field along a geodesic γ is a vector field along γ that satisfied the equation

$$J''(t) + R(J(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0$$

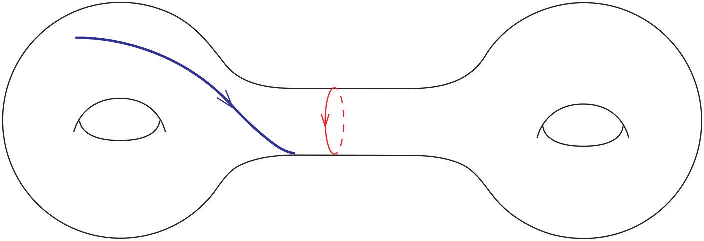
The rank of a vector $v \in T^1M$ is the dimension of the space of parallel Jacobi vector fields on the geodesic $\gamma_v : \mathbb{R} \to M$.

The rank of M is the minimum rank over all vectors in T^1M (always at least 1).

Standing Assumption: M is a compact rank 1 manifold with nonpositive curvature. (Rules out manifolds such as the torus.)

Example

singular geodesic
regular geodesic



Previous result

Reg = set of vectors in T^1M whose geodesics are regular, and
Sing= set of vectors in T^1M whose geodesics are singular.

$T^1M = \mathrm{Reg} \cup \mathrm{Sing}$

Reg is open and dense, and geodesic flow is ergodic on Reg.

Open problem: What is Liouville measure of Reg?

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Open problem: What is Liouville measure of Reg?

(Knieper, 98) There is a unique measure of maximal entropy. It is supported on Reg.

For surfaces $h_{top}(\mathrm{Sing}, \mathcal{F}) = 0$. Generally, $h_{top}(\mathrm{Sing}, \mathcal{F}) < h_{top}(\mathcal{F})$

Main results

Assume: M is a compact rank 1 manifold with nonpositive curvature and $\mathrm{Sing} \neq \emptyset$.

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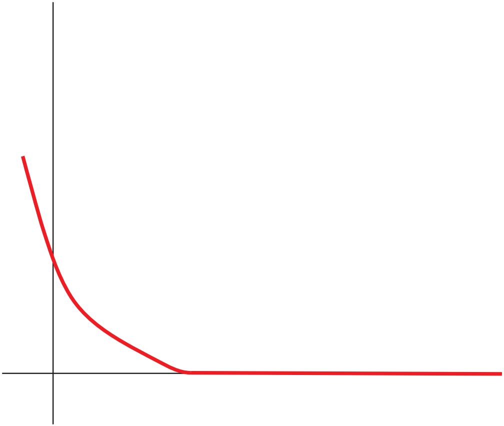
Theorem

(Burns, Climenhaga, F, Thompson) There is $q_0 > 0$ *such that if* $q \in (-q_0, q_0)$*, then the potential* $q\varphi^u$ *has a unique equilibrium state.*

$P(q\varphi^u)$ for a surface with Sing $\neq \emptyset$

The graph of $q \mapsto P(-q\varphi^u)$ has a corner at $(1, 0)$ created by μ_{Liou} and measures supported on Sing.



Outline

Bowen's result

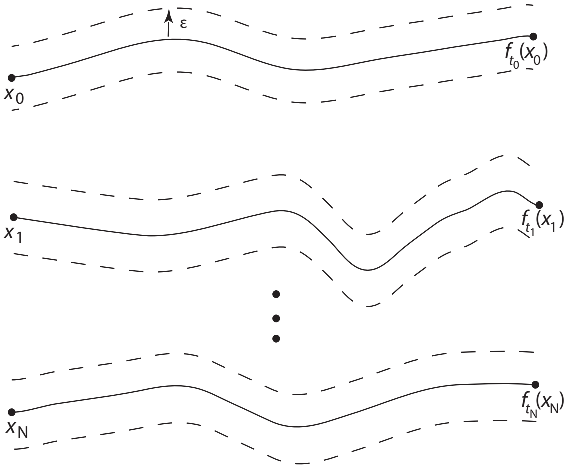
There is a unique equilibrium state for φ if

- $\mathcal{F}$ is expansive: For every $c > 0$ there is some $\epsilon > 0$ such that $d(f_t x, f_{s(t)} y) < \epsilon$ for all $t \in \mathbb{R}$ all $x, y \in \mathbb{R}$ and a continuous function $s : \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$ if $y = f_\gamma x$ for some $\gamma \in [-c, c]$.
- $\mathcal{F}$ has specification: next slide
- φ has the Bowen property: there exist $K > 0$ and $\epsilon > 0$ such that for any $T > 0$, if $d(f_t x, f_t y) < \epsilon$ for $0 \le t \le T$, then

$$\left| \int_0^T \varphi(f_t x) dt - \int_0^T \varphi(f_t y) dt \right| < K.$$

Specification

Given $\epsilon > 0 \ \exists \ \tau(\epsilon) > 0$ such that for all $(x_0, t_0), ..., (x_N, t_N) \in X \times [0, \infty)$ there exists a point y and $\tau_i \in [0, \tau(\epsilon)]$ for $0 \leq i < N$ such that



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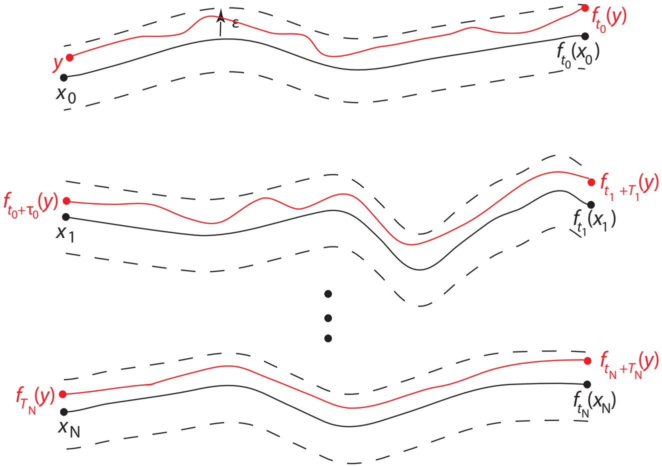


Climenhaga-Thompson Idea

Nonuniform version of Bowen's approach

Even if the flow may not be expansive, nor have specification, and the potential may not be Bowen, there is a unique equilibrium state if for large T the set of orbit segments of length at most T that have

- expansive properties
- specification for orbits segments of length at most T
- Bowen-like properties

has sufficiently large pressure, **and** the set of orbit segments of length T that don't have those properties has sufficiently small pressure

Decomposing orbit segments

$\mathcal{F} = f_t$ flow on X

$\mathcal{O} = X \times [0, \infty) = \{\text{finite orbit segments}\}$

$* =$ concatenation of orbit segments

Three subsets of $\mathcal{O}$: $\mathcal{P}$ (prefix), $\mathcal{G}$ (good), $\mathcal{S}$ (suffix)

So for each $(x, t) \in \mathcal{O}$ $\exists$ $p = p(x, t) \geq 0$, $g = g(x, t) \geq 0$, and $s = s(x, t) \geq 0$ such that

- $(x, p) \in \mathcal{P}$,
- $(f_p(x), g) \in \mathcal{G}$,
- $(f_{p+g}(x), s) \in \mathcal{S}$, and
- $p + g + s = t$.

Outline of theorem

Suppose we have a decomposition (so sets $\mathcal{P}, \mathcal{G}, \mathcal{S} \subset \mathcal{O}$ and functions p, g, and s):

- expansivity and specification for orbit segments in $\mathcal{G}$
- φ has the Bowen property on segments in $\mathcal{G}$
- $P(\varphi; \mathcal{P} \cup \mathcal{S}) < P(\varphi)$

Then φ has a unique equilibrium state

Some Previous results

Climenhaga-Thompson: (2012) symbolic systems such as β-shifts

Climenhaga-F-Thompson: (preprint) partially hyperbolic examples

Climenhaga-Thompson: (preprint) Flow version of the theorem

Outline

Decomposition for surfaces

$\kappa(v) =$ minimium curvature of the two horospheres othogonal to v

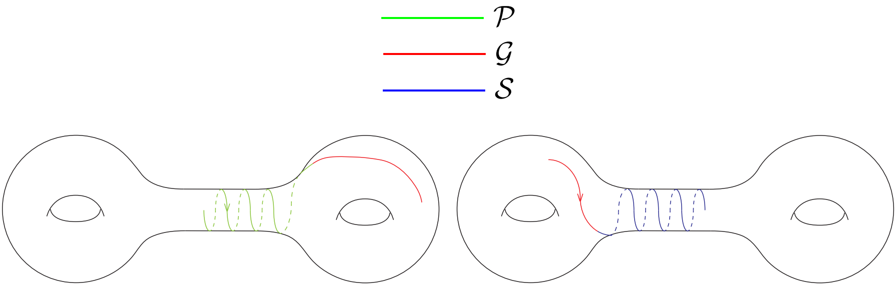
(v,t) is δ-bad if $\mathrm{Leb}\{s \in [0,t] : \kappa(f_s(v)) \geq \delta\} \leq \delta t$.
(less than δ - proportion of the time where we have δ curvature)

$\mathcal{P} = \mathcal{S} = \{(v,t) \in \mathcal{O} : (v,t) \text{ is } \delta\text{-bad}\}$

To decompose $(v,t) \in \mathcal{O}$:

- Find longest initial segment that is in $\mathcal{P}$
- Find longest tail segment of the remainder that is in $\mathcal{S}$
- What is left is in $\mathcal{G} = \mathcal{G}(\delta)$

Example of the decomposition



$(v,t) \in \mathcal{O}$ is decomposed to $(v,p) \in \mathcal{P}$, $(f_p(v), g) \in \mathcal{G}$, and $(f_{p+g}(v), s) \in \mathcal{S}$ where $p + g + s = n$.

Note: This does not tell us about the (forward or backward) asymptotic behavior of the orbit segments.

Properties of $\mathcal{G}$

Important property: For $(x, T) \in \mathcal{G}$ we know that for all $0 < t < T$ we have

- E^s_x is contracted a uniform amount (depending on δ) for Df_t and
- $E^u_{f_T(x)}$ is contracted a uniform amount for Df_{-t}.

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2. It is not known if φ^u is Hölder continuous (so Bowen) on T^1M. This has been a major obstacle with other techniques. We are able to show it is Bowen just on $\mathcal{G}$ and sidestep the problem.

Properties of $\mathcal{P}$ and $\mathcal{S}$

The idea is to show that the pressure on $\mathcal{P} \cup \mathcal{S}$ approaches the pressure Sing for δ small.

In fact we prove the next general result.

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Theorem

(Burns, Climenhaga, F, Thompson) If $\varphi : T^1M \to \mathbb{R}$ is continuous such that there exists some $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ the potential φ has the Bowen property on $\mathcal{G}(\delta)$, then there is a dichotomy:

- *either $P(\varphi, \mathrm{Sing}) < P(\varphi)$ and there is a unique equilibrium state, that is fully supported, or*
- *$P(\varphi, \mathrm{Sing}) = P(\varphi)$ and there is an equilibrium state supported on Sing.*

Thank You!